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Quantum Fluctuations in Thermal Vacuum State for Two LC Circuits With Mutual Inductance

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By means of thermal field dynamics theory, we study the quantum fluctuations of two LC circuits with mutual inductance at a finite temperature.

KEY WORDS: quantum fluctuation; thermal field dynamics; mesoscopic circuit.

1. INTRODUCTION

With the progress in nanotechnology and microelectronics, the trend in the miniaturization of integrated circuits and devices towards atomic scale dimensions becomes strong and definite. When the charge-carrier inelastic coherence length and the charge-carrier confinement dimension approach the Fermi wavelength, the classical physics is expected to be invalid and quantum effects must be taken into account. The quantum effects for a single LC lossless circuit were first discussed by Louisell (1973). Following a similar line of thought, many authors have discussed the quantum effects of mesoscopic circuits (Chen *et al.*, 1995; Fan *et al.*, 2000; Li *et al.*, 1996; Wang *et al.*, 2000; Xiao-Quang *et al.*, 1998; Zhang *et al.*, 1998). However, Louisell's result is obtained at $T = 0$; the Joule heat effects have not been taken into account. Electric currents in a circuit (not superconductors) produce Joule heat, and practical electric circuits are always working at a finite temperature. There is no doubt that the study of quantum noise of mesoscopic circuits at a finite temperature is very important both theoretically and experimentally. Recently, Fan Hong-yi and Pan Xiao-yin have studied the quantization of two LC circuits at the temperature $T = 0$ (Hong-yi *et al.*, 1998). In this paper, we employ thermal field dynamics (TFD) theory (Takahashi and Umezawa, 1975) to study the quantum fluctuations of the same system at a finite temperature.

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2. QUANTIZATION OF TWO LC CIRCUITS WITH MUTUAL INDUCTANCE

For a system including two LC circuits with mutual inductance, the classical Hamiltonian is given by Hong-yi *et al.* (1998)

$$H = \frac{1}{2A}\left(\frac{p_1^2}{l_1} + \frac{p_2^2}{l_2}\right) + \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} - \frac{m}{Al_1l_2}p_1p_2 - q_1\varepsilon_1(t) - q_2\varepsilon_2(t), \quad (1)$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are external sources, l_1 and l_2 are the self-inductance coefficients, m is the mutual-inductance coefficient between two circuits, c_1 and c_2 are the capacitances, $q_1 = q_1(t)$ and $q_2 = q_2(t)$ are the electrical charges of the two circuits, and their conjugate momenta p_1 and p_2 are, respectively,

$$p_1 = l_1\frac{dq_1}{dt} + m\frac{dq_2}{dt}, \quad (2)$$

$$p_2 = l_2\frac{dq_2}{dt} + m\frac{dq_1}{dt}, \quad (3)$$

$$A = 1 - \frac{m^2}{l_1l_2}. \quad (4)$$

It is obvious that Eq. (1) is analogous to two harmonic oscillators with a kinetic coupling term. According to the standard quantization principle, we quantize the system by identifying q_k and p_k as Hermite operators and imposing the commutation relation

$$[q_1, p_j] = i\delta_{ij}, \qquad (i, j = 1, 2) \quad (5)$$

(For convenience, here and henceforth we set Plank constant $h/2\pi = 1$.) Therefore Eq. (1) represents a pair of quantized harmonic oscillators with kinetic coupling. We now introduce the following unitary operator U to diagonalize H:

$$U = \int\int \left|u\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right| dq_1dq_2, \quad (6a)$$

where the state vector $|\binom{q_1}{q_2}\rangle$ in Eq. (6a) is the two-mode coordinate eigenstate, and

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (6b)$$

with

$$a = B\cos(\varphi/2), \quad b = B\sin(\varphi/2), \quad (6c)$$

$$c = -B^{-1}\sin(\varphi/2), \quad d = B^{-1}\cos(\varphi/2), \quad (6d)$$

$$B = \left(\frac{c_1}{c_2}\right)^{1/4}, \tag{6e}$$

$$tg\varphi = \frac{2m\sqrt{c_1 c_2}}{L_2 C_2 - L_1 C_1}. \tag{6f}$$

With the help of the technique of integration within an ordered product of operators (IWOP) (Hong-yi, 1997; Hong-yi *et al.*, 1987) we perform the integration in Eq. (6a) and obtain the normal product form of U (denoted by ::)

$$\begin{aligned} U &= \sec hr \exp\left[\frac{1}{2}\tanh r\left(a_1^{+2} - a_2^{+2}\right)\right] \\ &: \exp\left[(a_1^+, a_2^+)(G-1)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right] : \\ &\exp\left\{-\frac{1}{2}\tanh r\left[\left(a_2^2 - a_2^2\right)\cos\varphi + a_1 a_2 \sin\varphi\right]\right\}, \end{aligned} \tag{7}$$

where

$$G = \operatorname{sech} r \begin{pmatrix} \cos(\varphi/2) & \sin(\varphi/2) \\ -\sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix}, \tag{8a}$$

$$r = \frac{1}{4}\ln\left(\frac{c_1}{c_2}\right). \tag{8b}$$

From Eq. (6a) we can deduce

$$U^{-1} q_1 U = a q_1 + b q_2, \qquad U^{-1} q_2 U = c q_1 + d q_2, \tag{9}$$

$$U^{-1} p_1 U = d p_1 - c p_2, \qquad U^{-1} p_2 U = b p_1 + a p_2. \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (1), we obtain the diagonalized Hamiltonian H'

$$\begin{aligned} H' &= U^{-1} H U \\ &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{K_1}{2} q_1^2 + \frac{K_2}{2} q_2^2 + V_1(t) q_1 + V_2(t) q_2, \end{aligned} \tag{11}$$

where

$$\frac{1}{m_1} = \frac{1}{A l_1 l_2}\left(l_2 d^2 + l_1 b^2 + 2mbd\right), \tag{12}$$

$$\frac{1}{m_2} = \frac{1}{A l_1 l_2}\left(l_2 c^2 + l_1 a^2 + 2mac\right), \tag{13}$$

$$K_1 = \frac{a^2}{c_1} + \frac{c^2}{c_2}, \qquad K_2 = \frac{b^2}{c_1} + \frac{d^2}{c_2}, \tag{14}$$

$$V_1(t) = -a\varepsilon_1(t) - c\varepsilon_2(t), \qquad V_2(t) = -b\varepsilon_1(t) - d\varepsilon_2(t). \tag{15}$$

It is obvious that Eq. (11) is the Hamiltonians of two independent quantum harmonic oscillators with frequencies $\omega_j = \sqrt{K_j/m_j}$, $(j = 1, 2)$.
Letting

$$a_k = \sqrt{\frac{m_k\omega_k}{2}}\left(q_k + \frac{i}{m_k\omega_k}p_k\right), \qquad (k = 1, 2), \tag{16}$$

$$a_k^+ = \sqrt{\frac{m_k\omega_k}{2}}\left(q_k - \frac{i}{m_k\omega_k}p_k\right) \qquad (k = 1, 2). \tag{17}$$

Equation (11) can be rewritten as

$$H' = \omega_1\left(a_1^+a_1 + \frac{1}{2}\right) + \omega_2\left(a_2^+a_2 + \frac{1}{2}\right) + \gamma_1(t)(a_1 + a_1^+) + \gamma_2(t)(a_2 + a_2^+), \tag{18}$$

where

$$\gamma_k(t) = \frac{V_k(t)}{\sqrt{2m_k\omega_k}}, \qquad (k = 1, 2). \tag{19}$$

It can be proved that the time evolution operator $U_s(t, 0)$ corresponding to H' is given by Hong-yi (1997).

$$U_s(t, 0) = \exp\lfloor -i(\omega_1 a_1^+a_1 + \omega_2 a_2^+a_2)t\rfloor \exp\lfloor -i(\eta_1^* a_1^+ + \eta_1 a_1 + \eta_2^* a_2^* + \eta_2 a_2)\rfloor, \tag{20}$$

here we have omitted a phase factor,

$$\eta_k(t) = \int_0^l \gamma_k(\tau)\exp(-i\omega_k\tau)d\tau, \qquad (k = 1, 2). \tag{21}$$

Therefore, the wave function of the system at time t is given by

$$|\psi(t)\rangle = UU_s(t, 0)|\psi(0)\rangle, \tag{22}$$

here $|\psi(0)\rangle$ is the the wave function of the system at initial time $t = 0$.

If the the wave function of the system at initial time $t = 0$ is in the ground state $|00\rangle$, then we have

$$\begin{aligned} |\psi(t)\rangle &= UU_s(t, 0)|00\rangle \\ &= U\exp\left[-\frac{1}{2}(|\eta_1(t)|^2 + |\eta_2(t)|^2)\right]\exp\{-i[\eta_1^*(t)e^{-i\omega_1 t}a_1^+ \\ &\quad + \eta_2^*(t)e^{-i\omega_2 t}a_2^+]\}|00\rangle \\ &= U|z_1, z_2\rangle, \end{aligned} \tag{23}$$

where $|z_1z_2\rangle$ is two-mode coherent state with $z_k = -i\eta_k^*(t)e^{-i\omega_k t}$.

When the external electric source is only instantaneously switched on, say, for an infinitesimal time $t = \rho \to 0$ and then switched off, in this case, the system is in squeezed vacuum state:

$$|\psi(t=\rho)\rangle_{\rho\to 0} = U|00\rangle = \operatorname{sech} r\ \exp\left[\frac{1}{2}\tanh r\left(a_1^{+^2} - a_2^{+^2}\right)\right]|00\rangle. \quad (24)$$

It should be pointed that Eq. (24) is different from Eq. (20). Here, the ground state of the system is in squeezed vacuum state, not rotated squeezed vacuum state. The reason is that the unitary transformation operator U in Eq. (6a) is different from that in Hong-yi *et al.* (1998). Moreover, the squeezed parameter r only depends on the ratio of the capacitances of the two-component circuit (see Eq. (8b)).

3. QUANTUM FLUCTUATION OF THE SYSTEM

We now study the quantum fluctuation of the system at finite temperature. The effect of temperature can be introduced in terms of TFD theory, which was invented by Takahashi and Umezawa (1975). In TFD one associates (a, a^+), acting on a Hilbert space, with thermal freedom operators $(\tilde{a}, \tilde{a}^+)$ in the extended Hilbert space (a fictitious spac). The operators $\tilde{a}$ and $\tilde{a}^+$ obey the following commutation relation:

$$[\tilde{a}, \tilde{a}^+] = 1, \qquad [\tilde{a}, a] = [\tilde{a}, a^+] = 0 \quad (25)$$

The ensemble average of an operator A defined in the original Hilbert space,

$$\langle A\rangle = z^{-1}(\beta) tr(Ae^{-\beta H}), \qquad z(\beta) = tr(e^{-\beta H}), \quad (26)$$

can be calculated as a pure state (so-called thermal vacuum state) average, namely

$$\langle A\rangle = \langle 0(\beta)|A|0(\beta)\rangle, \quad (27)$$

where $\beta = (kT)^{-1}$ and k is the Boltzmann constant. Takahashi and Umezawa have proved that at finite temperature T the thermal vacuum state $|0(\beta)\rangle$ for a harmonic oscillator is given by Takahashi and Umezawa (1975).

$$|0(\beta)\rangle = S(\theta)|0\tilde{0}\rangle = \operatorname{sech} r\theta \exp(a^+\tilde{a}^+ \tanh\theta)|0\tilde{0}\rangle, \quad (28)$$

where the thermal operator $S(\theta)$ is

$$S(\theta) = \exp\lfloor -\theta(a\tilde{a} - a^+\tilde{a}^+)\rfloor, \quad (29)$$

$$\tanh\theta = \exp\left(-\frac{\omega}{2kT}\right). \quad (30)$$

We now suppose the initial state of the system is in two-mode thermal vacuum state

$$\begin{aligned}|\psi(0)\rangle &= |0(\beta)\rangle_1|0(\beta)\rangle_2 = S(\theta_1)S(\theta_2)|0\tilde{0}\rangle_1|0\tilde{0}\rangle_2\\ &= \operatorname{sech} r\theta_1 \sec\ h\theta_2 \exp(a_1^+\tilde{a}_1^+ \tanh\theta_1 + a_2^+\tilde{a}_2^+ \tanh\theta_2)|0\tilde{0}\rangle_1|0\tilde{0}\rangle_2, \quad (31)\end{aligned}$$

where

$$S(\theta_k) = \exp[-\theta_k(a_k\tilde{a}_k - a_k^+\tilde{a}_k^+)], \qquad (k = 1, 2), \tag{32}$$

$$\tanh\theta_k = \exp\left(-\frac{h\omega_k}{2kT}\right), \qquad (k = 1, 2). \tag{33}$$

Then we have

$$|\psi(t)\rangle = UU_s(t, 0)|0(\beta)\rangle_1|0(\beta)\rangle_2. \tag{34}$$

It can be proved that

$$S^{-1}(\theta_1)a_1S(\theta_1) = a_1\cosh\theta_1 + \tilde{a}_1^+\sinh\theta_1, \tag{35a}$$

$$S^{-1}(\theta_2)a_2S(\theta_2) = a_2\cosh\theta_2 + \tilde{a}_2^+\sinh\theta_2, \tag{35b}$$

$$U_s^{-1}a_1U = a_1e^{-i\omega_1 t} + z_1, \qquad U_s^{-1}a_2U = a_2e^{-1\omega_2 t} + z_2, \tag{35c}$$

where

$$z_k = -i\eta_k^*(t)e^{-i\omega_k t} \tag{36}$$

From Eqs. (9) (10), and (35) we have

$$\langle q_1\rangle = \sqrt{\frac{2}{\mu\omega_1}}a\mathrm{Re}(z_1) + \sqrt{\frac{2}{\mu\omega_2}}b\mathrm{Re}(z_2), \tag{37}$$

$$\langle q_2\rangle = \sqrt{\frac{2}{\mu\omega_1}}c\mathrm{Re}(z_1) + \sqrt{\frac{2}{\mu\omega_2}}d\mathrm{Re}(z_2), \tag{38}$$

$$\langle p_1\rangle = \sqrt{2\mu\omega_1}d\mathrm{Im}(z_1) - \sqrt{2\mu\omega_2}c\mathrm{Im}(z_2), \tag{39}$$

$$\langle p_2\rangle = -\sqrt{2\mu\omega_1}b\mathrm{Im}(z_1) + \sqrt{2\mu\omega_2}a\mathrm{Im}(z_2), \tag{40}$$

$$\langle q_1^2\rangle = \frac{a^2}{2\mu\omega_1}[4\mathrm{Re}(z_1) + \cosh(2\theta_1)] + \frac{b^2}{2\mu\omega_2}[4\mathrm{Re}^2(z_2) + \cosh(2\theta_2)] + \frac{4ab}{\mu\sqrt{\omega_1\omega_2}}\mathrm{Re}(z_1)\mathrm{Re}(z_2), \tag{41}$$

$$\langle q_2^2\rangle = \frac{c^2}{2\mu\omega_1}[4\mathrm{Re}^2(z_1) + \cosh(2\theta_1)] + \frac{d^2}{2\mu\omega_2}[4\mathrm{Re}^2(z_2) + \cosh(2\theta_2)] + \frac{4cd}{\mu\sqrt{\omega_1\omega_2}}\mathrm{Re}(z_1)\mathrm{Re}(z_2), \tag{42}$$

$$\langle p_1^2\rangle = \frac{1}{2}d^2\mu\omega_1[4\mathrm{Im}^2(z_1) + \cosh(2\theta_1)] + \frac{1}{2}c^2\mu\omega_2[4\mathrm{Im}^2(z_2) + \cosh(2\theta_2)] - 4cd\mu\sqrt{\omega_1\omega_1}\mathrm{Im}(z_1)\mathrm{Im}(z_2), \tag{43}$$

$$\langle p_2^2 \rangle = \frac{1}{2} b^2 \mu\omega_1 [4\mathrm{Im}^2(z_1) + \cosh(2\theta_1)] + \frac{1}{2} a^2 \mu\omega_2 [4\mathrm{Im}^2(z_2) + \cosh(2\theta_2)] - 4ab\mu\sqrt{\omega_1\omega_2}\mathrm{Im}(z_1)\mathrm{Im}(z_2). \tag{44}$$

Therefore, the fluctuations of the charges q_j and their conjugate variables p_j are, respectively,

$$\langle(\Delta q_1)^2\rangle = \langle q_1^2\rangle - \langle q_1\rangle^2 = \frac{a^2}{2\mu\omega_1}\cosh(2\theta_1) + \frac{b^2}{2\mu\omega_2}\cosh(2\theta_2), \tag{45}$$

$$\langle(\Delta q_2)^2\rangle = \langle q_2^2\rangle - \langle q_2\rangle^2 = \frac{c^2}{2\mu\omega_1}\cosh(2\theta_1) + \frac{d^2}{2\mu\omega_2}\cosh(2\theta_2), \tag{46}$$

$$\langle(\Delta p_1)^2\rangle = \langle p_1^2\rangle - \langle p_1\rangle^2 = \frac{1}{2} d^2 \mu\omega_1 \cosh(2\theta_1) + \frac{1}{2} c^2 \mu\omega_2 \cosh(2\theta_2), \tag{47}$$

$$\langle(\Delta p_2)^2\rangle = \langle p_2^2\rangle - \langle p_2\rangle^2 = \frac{1}{2} b^2 \mu\omega_1 \cosh(2\theta_1) + \frac{1}{2} a^2 \mu\omega_2 \cosh(2\theta_2) \tag{48}$$

From Eqs. (37) to (48), we can draw the following conclusion. (1) The averages of the charges q_j and their conjugate variables p_j only depend on parameters z_k and have nothing to do with temperature T. (2) The mean-square values of the charges q_j and their conjugate variables p_j are dependent on both the parameters z_k and temperature T. (3) The quantum fluctuations of the charges q_j and their conjugate variables p_j are independent on the parameters z_k. Since the parameters z_k are related to the source $\varepsilon_1(t)$ and $\varepsilon_2(t)$ by Eq. (36); therefore, we can obtain the conclusion that the quantum fluctuations of the charges q_j and their conjugate variables p_j have nothing to do with the concrete forms of the sources $\varepsilon_1(t)$ and $\varepsilon_2(t)$. (4) At a finite temperature the system exhibits more quantum noise than that at the $T \to 0$ case.

4. CONCLUSION AND DISCUSSION

In short, we have studied the quantization of two LC circuit with mutual inductance at a finite temperature and discussed its time evolution. When the external electric source is only instantaneously switched on, say, for an infinitesimal time $t = \rho \to 0$ and then switched off, the ground state of the system is in squeezed vacuum state and the squeezed parameter r only depends on the ratio of the capacitances of the two-component circuit. By means of TFD theory, we have studied the quantum fluctuation of the system at a finite temperature. The results show that at a finite temperature the system exhibits more quantum noise than that at the $T \to 0$ case. It is remarkable the quantum fluctuations of the charges q_j and their conjugate variables p_j have nothing to do with the concrete form of the source $\varepsilon(t)$.

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